CONCISE RULES

For Computing the Effects of

REFRACTION and PARALLAX

IN

Varying the Apparent Diſtance of the
Moon from the Sun or a Star;

ALSO

An eaſy Rule of Approximation for computing the
Diſtance of the Moon from a Star,

The Longitudes and Latitudes of both being given, with
Demonſtrations of the ſame:

By the Rev. NEVIL MASKELYNE, A.M.
Fellow of Trinity College, in the Univerſity of Cambridge, F.R.S.

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CONCISE RULES

For Computing the Effects of

REFRACTION and PARALLAX

IN

Varying the Apparent Diſtance of the Moon from the Sun or a Star, &c.

THE following rules, excepting one, are the ſame which I have already communicated to the Royal Society, but without demonſtration, in a letter to the reverend Dr. Birch from St. Helena, containing the reſults of my obſervations of the diſtance of the Moon from the Sun and fixed ſtars, taken in my voyage thither, for finding the longitude of the ſhip from time to time; ſince printed in Part II. Vol. LII. of the Philoſophical Tranſactions for 1762. The two rules for the correction of refraction and parallax I have alſo ſince communicated to the public in my Britiſh Mariner's Guideto the diſcovery of longitude from like obſervations of the Moon; and have added in the Preface a rule for computing a ſecond but ſmaller correction of parrallax, neceſſary on account of a ſmall

imperfection

imperfection lying in the firſt rule derived from the fluxions of a ſpherical triangle. To the rules I have here ſubjoined their demonſtrations.

With reſpect to the uſefulneſs of theſe rules, I cannot but entertain hopes that they will appear more ſimple and eaſy than any yet propoſed for the ſame purpoſe: the laſt rule, for computing the diſtance of the Moon from a ſtar, though only an approximation, being ſo very exact, ſeems particularly adapted for the conſtruction of a nautical Ephemeris, containing the diſtances of the Moon from the Sun and proper fixed ſtars ready calculated for the purpoſe of finding the longitude from obſervations of the Moon at ſea; an aſſiſtance which, in an age abounding with ſo many able computers, mariners need not doubt they will be provided with, as ſoon as they manifeſt a proper diſpoſition to make uſe of it.

A RULE

To compute the contraction of the apparent diſtance of any two heavenly bodies by refraction; the zenith diſtances of both, and their diſtance from each other being given nearly.

Add together the tangents of half the ſum, and half the difference of the zenithdiſtances; their ſum, abating 10 from the index, is the tangent of arc the firſt. To the tangent of arc the firſt, juſt found, add the co-tangent of half the diſtance of the ſtars; the ſum, abating 10 from the index, is the tangent of arc the ſecond. Then add together the tangent of double the firſt arc, the co-ſecant of double the double the ſecond arch, and the conſtant logarithm of 114″ or 2,0569: the ſum abating 20 from the index, is the logarithm of the number of ſeconds required, by which the diſtance of the ſtars is contracted by refraction: which therefore added to the obſerved diſtance gives the true diſtance cleared from the effect of refraction.

Explication of the foundation of the preceding rule.

This rule is founded upon an hypotheſis that the refraction in altitude is as the tangent of the zenith diſtance: and the refraction at the altitude of 45 degrees being 57″, according to Dr. Bradley's obſervations, therefore the refraction at any altitude, calling the radius unity is = 57″ × tangent of the zenith diſtance. This rule is exact enough for the purpoſe of the calculation of the longitude from obſervations of the diſtance of the Moon from ſtars at ſea as low down as the altitude of 10°, for there the error is only 10″ from the truth. But, if the altitude of the Moon or ſtar be leſs than 10°, the rule may be ſtill made to anſwer ſufficiently, by only firſt correcting the obſerved zenith diſtances by ſubſtracting from them three times the refraction correſponding to them taken out of any common table of refraction, and making the computation with the zenith diſtances thus corrected. This correction depends upon Dr. Bradley's rule for refraction, which he found to anſwer, in a manner exactly, from the zenith quite down to the horiſon, namely that the refraction is = 57″ × tangent of the apparent zenith diſtance leſſened by three times the correſponding refraction taken out of any common table.

Demonſtration of the preceding rule.

Let ZXY Tab. XVII. Fig. 1. repreſent a ſpherical triangle formed by great circles joining the zenith Z and the ſtars X and Y. Refraction acting in the vertical circles ZX and ZY will carry the ſtar X nearer the zenith by a quantity $Xb = 57'' \times$ tangent of ZX, and the ſtars Y towards Z by the quantity $Yd = 57'' \times$ tangent of ZY; ſo that the apparent diſtance of the two ſtars will be *bd* inſtead of XY or leſs than XY, the true diſtance, by the Sun of the two little ſpaces X*a*, Y*e*, terminated by the perpendiculars *ba* and *de*. The little ſpace $Xa = Xb \times$ coſine of the angle ZXY (calling radius unity) $= 57'' \times$ tang. of ZX $\times$ coſine of angle ZXY, or, by ſpherics, $= 57'' \times$ tang. of XP. (ZP being an arch drawn from Z perpendicular to the arch XY). In like manner the little ſpace $Ye = 57'' \times$ tang. of YP; and therefore $Xa + Ye$ or the total effect of refraction $= 57'' \times \overline{\text{tang. } XP + \text{tang. } YP}$. Let M be the middle of the arch XY, and put the tangent of XM or YM or $\frac{1}{2}XY = t$, and the tangent of MP, or the diſtance of the perpendicular ZP from M the middle of the arch $XY = n$. By trigonometry, tang. of XP or $XM + MP = \frac{t+n}{1-tn}$, and tang. of YP or $YM - MP = \frac{t-n}{1+tn}$; the ſum of which, or tang. $XP +$ tang. $YP = \frac{t+n}{1-tn} + \frac{t-n}{1+tn} = \frac{2t+2tn^2}{1-t^2n^2} = \frac{2tn}{1-t^2n^2} \times \frac{1+n^2}{2n} \times 2$. Now

2*tn*

$\frac{2tn}{1-t^2n^2}$ is the tangent of double the angle whoſe tangent is tn, and tn or the product of the tangents of $\frac{1}{2}$ XY and MP, by ſpherics, is equal to the product of the tangents of half the ſum and half the difference of the zenith diſtances ZX and ZY; whence $\frac{2tn}{1-t^2n^2}$ is equal to the tangent of double arch the firſt found by the rule. Alſo arch the ſecond found by the rule being by ſpherics = MP, whoſe tangent is repreſented by n, and $\frac{1+n^2}{2n}$ being by trigonometry equal to the coſecant of double the arch whoſe tangent is n, therefore $\frac{1+n^2}{2n}$ = coſecant of 2 MP or double arch the ſecond. Whence the rule is manifeſt; namely, that Xa + Ye, the total effect of refraction in contracting the apparent diſtance of the two ſtars $= 57'' \times \overline{\text{tan. } XP + \text{tan. } YP}$ = double 57″ or 114″ × tang. of double the firſt arch × coſecant of double the ſecond arch. Q. E. D.

REMARK.

When the perpendicular arch ZP, let fall from the Zenith on the arch XY, falls without the triangle ZXY, the effect of refraction in diminiſhing the apparent diſtance of the ſtars X, Y is the difference of Xa and Ye: but the rule being general, gives always the ſum or difference, which-ever it be, which is a great advantage, and removes all grounds of ambiguity in the correction of refraction; as the the total effect thereof always diminiſhes the diſtance of two ſtars from each other, however they are poſited.

A RULE

To compute the contraction or augmentation of the apparent diſtance of the Moon from a ſtar, on account of the Moon's parallax; the zenith diſtances of the Moon and ſtar, and their diſtance from each other being given nearly.

Add together the tangents of half the ſum, and half the difference of the zenith diſtances of the Moon and ſtar, and the cotangent of half the diſtance of the Moon from the ſtar; the ſum, abating 20 from the index, is the tangent of an arch, which call A. Then, if the zenith diſtance of the Moon is greater than that of the ſtar, take the Sum of the arch A, juſt found, and half the diſtance of the Moon from the ſtar; but, if the zenith diſtance of the Moon be leſs than that of the ſtar, take the difference of the ſaid arch A and half the diſtance of the Moon from the ſtar; and the ſum, or difference call B. To the tangent of B, thus found, add the coſine of the Moon's zenith diſtance, and the logarithm of the Moon's horizontal parallax, expreſſed in minutes and decimals; the ſum, abating 20 from the index, is the logarithm of the effect of parallax, tending always to augment the apparent diſtance of the Moon from the ſtar; except the zenith diſtance of the Moon be leſs than that of the ſtar, and, at the ſame time, the arch A be greater than half the diſtance of the Moon from the ſtar, in which caſe the effect of parallax diminiſhes the apparent diſtance of the Moon from the ſtar.

DEMONSTRATION.

In the ſpherical triangle ZLS, ſee Fig. 2, 3, 4, and 5th, Z repreſents the zenith, L the Moon, and S the ſtar; the effect of parallax depreſſing the Moon from L to r, r is the apparent place of the Moon, and rS the apparent diſtance of the Moon from the ſtar; let fall the perpendicular Lt upon rS, produced if neceſſary, and rt will be the difference of LS and rS, or the effect of parallax. Draw the arch ZP perpendicular to to rS, and let M be the middle of rS. The Moon's parallax in altitude, being to her horizontal parallax, as the ſine of her apparent zenith diſtance, to the radius, $Lr =$ Moon's horizontal parallax $\times$ ſine of Zr; and rt the effect of parallax upon the apparent diſtance of the Moon from the ſtar will be $= Lr \times$ coſ. $ZrS =$ horizontal parallax $\times$ ſin. $Zr \times$ coſ. ZrS (or, becauſe tan. rP : coſ. ZrP :: tan Zr : rad :: ſin Zr : cos Zr; and therefore ſin. $Zr \times$ cos $ZrS =$ coſ. $Zr \times$ tan rP) $=$ horizontal parallax $\times$ coſ. $Zr \times$ tan. rP agreeably to the rule. For it is evident by ſpherics that the arch A, found by the rule, is the ſame with MP the diſtance of the perpendicular from the middle of the arch rS: and it is evident, by the inſpection of the figures, that the arch B or rP is equal to the ſum of rM and MP, if the zenith diſtance of the Moon be greater than that of the ſtar, as in Fig. 2d and 4th; but is the difference of rM and MP, if the zenith diſtance of the Moon be leſs than that of the ſtar, as in Fig. 3d and 5th. Laſtly, it may appear from the

conſideration

conſideration of the figures, that, as the effect of parallax depreſſes the Moon directly towards the horizon, ſo it will always encreaſe her apparent diſtance from a ſtar, except in the caſe repreſented by Fig. 5th; that is to ſay, unleſs the zenith diſtance of the Moon be leſs than that of the ſtar, and, at the ſame time, the arch MP be greater than *r*M or half the diſtance of the Moon from the ſtar. Q. E. D.

Remarks on the uſe of the two foregoing rules.

It has been remarked, after the rule for refraction above, that if the altitudes of the Moon or ſtar are under 10 degrees, the zenith diſtances muſt be firſt leſſened by 3 times the refractions correſponding to their reſpective altitudes before the effect of refraction be computed.

But in order to compute the effect of parallax from the ſecond rule, the obſerved diſtance of the Moon from the ſtar muſt be firſt corrected by adding the effect of refraction to it found by rule the firſt; as muſt the obſerved altitudes of the Moon and ſtar be alſo corrected by taking from them their reſpective refraction in altitude, and the corrected arches thus found muſt be made uſe of in computing the parallax. Only, if the altitudes of the Moon and ſtar are both 10 degrees or more, part of the calculation of rule the ſecond may be ſaved, and arch the ſecond, found by rule the firſt, taken for arch A in the ſecond rule without any ſenſible error. In this caſe, it will be moſt convenient to obſerve the following order of computation inſtead of that before preſcribed to be uſed when the altitudes are under 10 degrees.

1ſt. Making uſe of the apparent altitudes of the Moon and ſtar uncorrected, compute arches the firſt and ſecond by the directions contained in the rule of refraction.

2dly. Taking arch the ſecond for arch A in the rule of parallax, compute the effect of parallax according to rule the ſecond.

3dly. With arches the firſt and ſecond compute the effect of refraction by rule the firſt.

4thly, and laſtly. Applying the two corrections of parallax and refraction duly, according to the rules, to the obſerved diſtance of the Moon from the ſtar, you will have the true and correct diſtance of the Moon from the ſtar, cleared both of refraction and parallax.

A RULE

For computing a ſecond, but ſmaller correction than the firſt, neceſſary to be applied to the obſervations of the diſtance of the Moon from a ſtar on account of parallax.

Call the principal effect of parallax, found by the preceding rule, the parallax in diſtance; and find the parallax anſwering to the Moon's altitude. Then to the conſtant logarithm 0.941 add the logarithm of the ſum of the parallax in altitude and the parallax in diſtance, the logarithm of the difference of the ſame parallaxes, and the cotangent of the obſerved diſtance of the Moon from the ſtar (corrected for refraction, and the principal effect of parallax), the ſum, abating 13 from the index, is the logarithm

logarithm of the number of ſeconds required, being the ſecond correction of parallax; and is always to be added to the diſtance of the Moon from the ſtar, firſt corrected for refraction, and the principal effect of parallax found above, in order to obtain the true diſtance; unleſs the diſtance exceeds 90 degrees, in which caſe it is to be ſubſtracted.

DEMONSTRATION.

Let L *Fig.* 6. repreſent the Moon's true place in the ſphere, and *r* her apparent place as depreſſed by parallax, S the place of the ſtar, and L *t* a perpendicular let fall from the true place of the Moon L upon the great circle *r* S joining the ſtar S and the apparent place of the Moon *r*; (all as in the four figures belonging to the preceding rule). Let L *a* be the arch of a parallel circle deſcribed from the ſtar S as a pole through the true place of the Moon L. S *a* terminated by the parallel circle L *a*, and not S *t* terminated by the perpendicular L *t*, as was ſuppoſed in the former demonſtration, is equal to S L or the true diſtance of the Moon from the ſtar, which was therefore computed too ſmall from the former rule by the little ſpace *a t*. Let L T and *a* T be the equal tangents of the equal arches L S and *a* S in L and *a*, meeting in the radius C S, drawn from the centre of the ſphere C and produced, in T. The ſpace L *a t*, on account of its ſmallneſs, may be looked upon as lying all in one plane namely L *a* T, and L *a* as the ſmall arch of a circle deſcribed from the point T as a centre with the line L T as a radius, through L and a, Lt as the ſine, and at as the verſe-ſine of the arch La; and conſequently at equal to the ſquare of Lt divided by $2\,LT$. But, the triangle Lrt being right-angled in t, the ſquare of Lt is equal to the difference of the ſquares of Lr and rt, and conſequently to the product of their ſum and difference; that is to ſay, $at = \frac{\overline{Lr+rt}\times\overline{Lr-rt}}{2\,LT}$ or (becauſe the tangent TL is equal to the ſquare of the radius CS divided by the cotangent of LS) $= \overline{Lr+rt}\times\overline{Lr-rt}\,\frac{\times\text{ cotangent of } LS}{2\times\text{ſquare of } CS}$. Now ſuppoſe the ſpaces Lr, rt to be expreſſed in minutes, which will be moſt convenient in practice, then the radius of the ſphere CS muſt be taken equal to $3437\frac{3}{4}$, for ſo many minutes are contained in an arch of a circle equal to its radius: and at will be $= \frac{\overline{Lr+rt}+\overline{Lr-rt}\times\text{ cotan. of } LS}{2\times 3437\frac{3}{4}\times 3437\frac{3}{4}}$. But, the contangents of ſimilar arches of circles of different radii being directly as the radii, therefore the cotangent of LS to the radius CS or $3437\frac{3}{4}$, is to the cotangent of the ſame arch to 10000000000, which is the radius to which the logarithmic tables are adapted, its logarithm being 10; as $3437\frac{3}{4}$ to 10000000000. Therefore the cotangent of $LS =$ tabular cotangent $\frac{\text{of } LS\times 3437\frac{3}{4}}{10000000000}$, which, being ſubſtituted in the value of at above, gives at, expreſſed in minutes $= \frac{\overline{Lr+rt}+\overline{Lr-rt}\times\text{ tabular cotangent } LS}{68755000000000}$; or, multiplying by 60, the value of at will come out in ſe-

conds $= \frac{\overline{Lr+rt} \times \overline{Lr-rt} \times \text{tabular cotangent of } LS}{1146000000000}$.
The logarithm of the denominator 1146000000000 is 12,059, inſtead of ſubſtracting which, when the operation is performed by logarithms, add 0,941 (its compliment to 13) to the value of the numerator found in logarithms, and ſubſtract 13 from the index: the remainder will be the value of *at* in ſeconds. Q. E. D.

A conciſe rule to find the diſtance of the Moon from a zodiacal ſtar, very nearly; the difference of the longitudes of the Moon and ſtar, and the latitudes of both being given.

To the coſine of the difference of the longitudes add the coſine of the difference of the latitudes, if both of the ſame denomination, or ſum; if of contrary denominations, the ſum of the two logarithms, abating 10 from the index, is the coſine of the approximate diſtance. This gives the true diſtance of the Moon from the Sun, being then nothing more than the common rule for finding the hypothenuſe of a right-angled ſpherical triangle from the two ſides given. But in the caſe of a zodiacal ſtar apply the following correction to the approximate diſtance thus found.

To the conſtant logarithm 5.3144 add the ſine of the Moon's latitude, the ſine of the ſtar's latitude, the verſe-ſine of the difference of longitude, and the coſecant of the approximate diſtance; the ſum of theſe 5 logarithms, abating 40 from the index, is the logarithm of a number of ſeconds, which ſubſtracted

ſtracted from the approximate diſtance, found before, if the latitudes of the moon and ſtar are of the ſame denomination, or added thereto, if they are of different denominations, gives the true diſtance of the Moon from the ſtar.

N. B. This rule, though only an approximation, is ſo very exact, that even, if the latitude of the Moon was 5°, and that of the ſtar 15°, the error would be only $10''$; and if the latitude of the Moon be 5°, and that of the ſtar 10°, the error is only $4''\frac{1}{2}$; and if the latitudes be leſs, will be leſs in proportion as the ſquares of the ſines of the latitudes decreaſe.

DEMONSTRATION.

Let P [Fig. 7.] repreſent one of the poles of the eclip- and Q, R the places of the Moon and ſtar. From R let the arch of a great circle RD be drawn perpendicular to PQ. By ſpherics, the tangent of $PD =$ tangent of $PR \times$ coſine of the angle RPD. And, by trigonometry, coſine of QD or $(QP - PD) = \text{coſ. } QP \times \text{coſ. } PD + \text{ſin. } QP \times \text{ſin. } PD = \text{coſ. } QP \times \text{coſ. } PD + \text{ſin. } QP \times \text{coſ. } PD \times \text{tan. } PD = \text{coſ. } PD \times \overline{\text{coſ. } QP + \text{ſin. } QP \times \text{tan. } PD} = \text{coſ. } PD \times \overline{\text{coſ. } QP + \text{ſin. } QP \times \text{tan. } PR \times \text{coſ. } P}$ $\therefore \text{coſ. } QD : \text{coſ. } PD :: \text{coſ. } QP + \text{ſin. } QP \times \text{tan. } PR \times \text{coſ. } P : 1$. But, by ſpherics, $\text{coſ. } QD : \text{coſ. } PD :: \text{coſ. } RQ : \text{coſ. } PR \therefore \text{coſ. } RQ : \text{coſ. } PR :: \text{coſ. } QP + \text{ſin. } QP \times \text{tan. } PR \times \text{coſ. } P : 1$. Whence $\text{coſ. } RQ = \text{coſ. } PR \times \text{coſ. } QP + \text{ſin. } QP \times \text{ſin. } PR \times \text{coſ. } P$: Now, by trigonometry, $\text{coſ. } (QP - PR)$

$= \text{coſ.}$

= coſ. Q P × coſ. P R + ſin. Q P × ſin. P R; whence ſin. Q P × ſin. P R = coſ. (Q P — P R) — coſ. Q P × coſ. P R; which being ſubſtituted above, gives coſ. R Q = coſ. (Q P — P R) × coſ. P — coſ. P R × coſ. Q P × coſ. P + coſ. P R × coſ. Q P = coſ. (Q P — P R) × coſ. P + verſe-ſine P × coſ. P R × coſ. Q P. Now put coſ. (Q P — P R) × coſ. P = coſ. G, or the approximate diſtance, then coſ. R Q — coſ. G, or (becauſe the difference of R Q and G is but ſmall) $\overline{G - RQ} \times \text{ſin.}\left(\frac{G+RQ}{2}\right)$ = verſe-ſine P × coſ. P R × coſ. P Q nearly. Whence

$$RQ = G - \frac{\text{verſe-ſine } P \times \text{coſ. } PR \times \text{coſ } QP}{\text{ſin. } \frac{G}{2} + \frac{RQ}{2}}$$

nearly $= G - \frac{\text{v. ſ. } P \times \text{coſ. } PR \times \text{coſ. } QP}{\text{ſin. } G}$ nearly.

Q. E. D.

Note, the error of this formula ariſes from taking $G = \frac{G + RQ}{2}$ by which means it will always give R Q too great, nearly by the following quantity, $\frac{1}{2}$ S *q* coſ. P R × S *q* coſ. Q P × cot. G × S *q* tan. $\frac{1}{2}$ G. This comes to a maximum when G is 60°, and is then $= \frac{1}{6}\sqrt{\frac{1}{3}}$ × S *q* coſ. R P × S *q* coſ. P Q. If the latitudes of the Moon and ſtar are both 5° it is $= 1''$. If the Moon's latitude be 5°, and that of the ſtar 10°, it is $= 4''\frac{1}{2}$; and if the latitude of the ſtar be 15° it is $= 10''$.

THE END.

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Fig. 1.

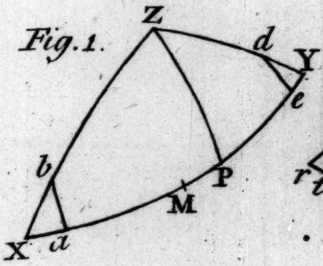


Fig. 2.

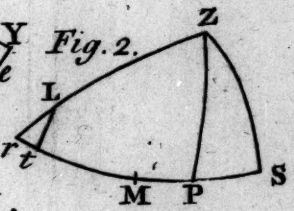


Fig. 3.

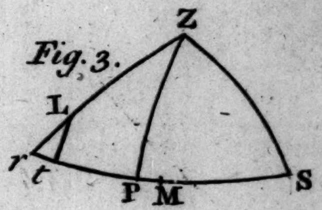


Fig. 4.

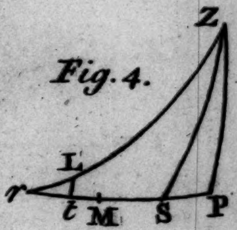


Fig. 5.

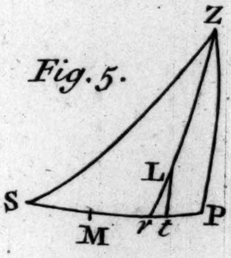


Fig. 6.

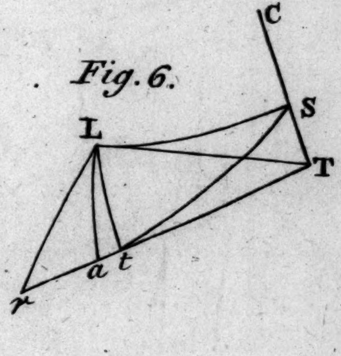
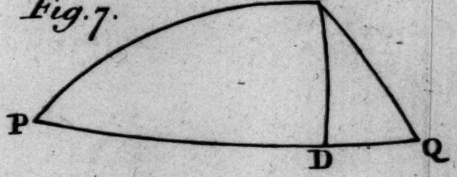


Fig. 7.



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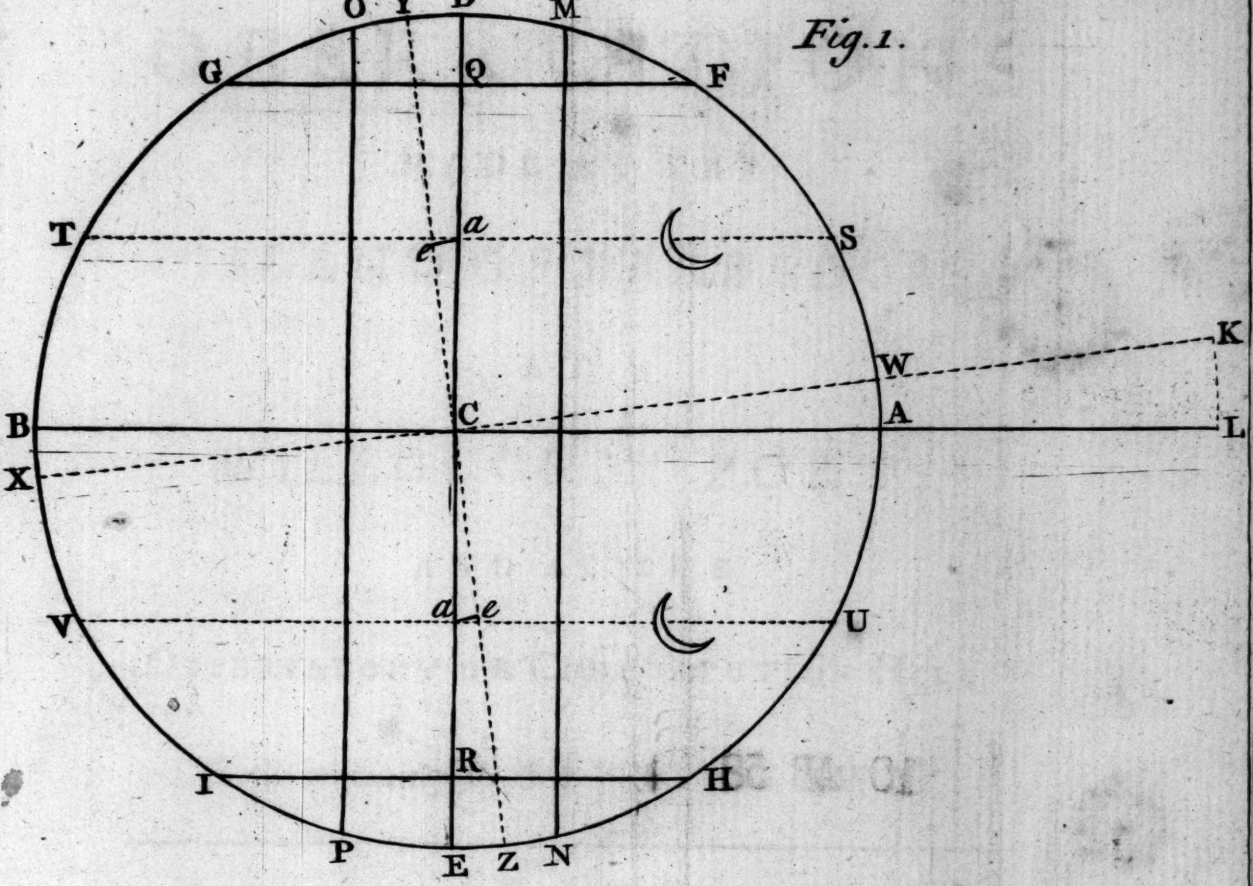

Fig. 1.

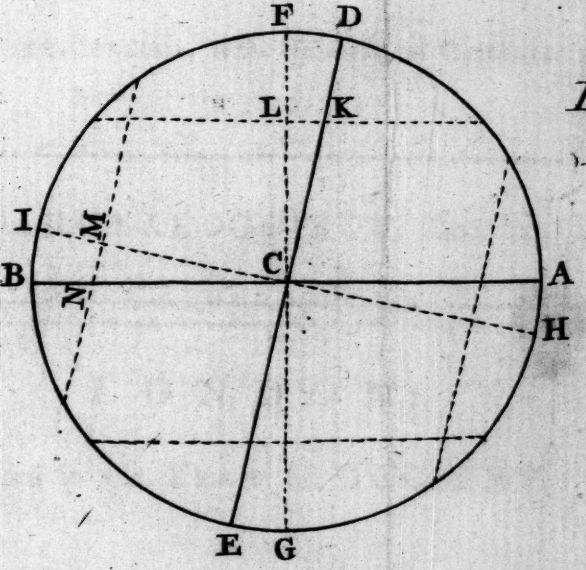

Fig. 2.

J. Mynde sc.